# Instability of the flow of two immiscible liquids with different viscosities in a pipe

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We study the flow of two immiscible fluids of different viscosities and equal density through a pipe under a pressure gradient. This problem has a continuum of solutions corresponding to arbitrarily prescribed interface shapes. The question therefore arises as to which of these solutions are stable and thus observable. Experiments have shown a tendency for the thinner fluid to encapsulate the thicker one. This has been 'explained' by the viscous-dissipation principle, which postulates that the amount of viscous dissipation is minimized for a given flow rate. For a circular pipe, this predicts a concentric configuration with the more viscous fluid located at the core. A linear stability analysis, which is carried out numerically, shows that while this configuration is stable when the more viscous fluid occupies most of the pipe, it is not stable when there is more of the thin fluid. Therefore the dissipation principle does not always hold, and the volume ratio is a crucial factor.

## 1. Introduction

The flow we consider is Hagen–Poiseuille flow in a cylindrical pipe of infinite length in which there are two fluids. The fluids are immiscible and have the same density but different viscosities. The flow is steady, purely axial and driven by a prescribed pressure gradient.

The equations governing the flow are the steady Navier–Stokes equations with the velocity and the pressure gradient in the axial direction, and incompressibility. The boundary conditions are: no slip at the pipe wall, and, at the unknown interface of the two fluids, the normal and shear stresses and the velocity are to be continuous. We specify the ratio of the cross-sectional area occupied by each fluid and study the problem of selection of the interface shapes using linear stability theory.

Theoretically, it is known that, if there is no surface tension, every interface position is allowed by the equations. If there is surface tension, then the interface must be circles or circular arcs terminating at the pipe wall. The number of possible steady solutions is still infinite. For example, if fluid 1 occupies $\frac{1}{3}$ of the cross-sectional area and fluid 2 occupies $\frac{2}{3}$ of the area, then 2 possible arrangements are shown in figure 1. Such non-uniqueness appears in the theory of steady two-fluid flows for many kinds of flow regimes (Joseph, Nguyen & Beavers 1984). On the other hand, experiments with the pipe flow indicate that whatever the initial configuration, the low-viscosity liquid will eventually encapsulate the thicker fluid. The encapsulation property has
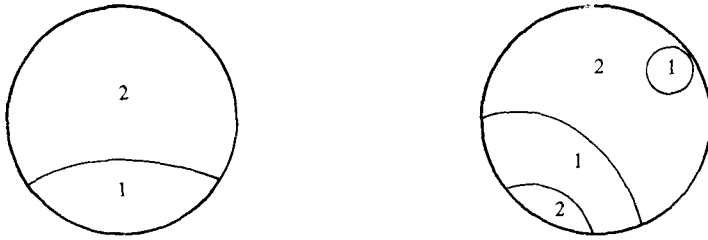
FIGURE 1

been observed for both high- and low-Reynolds-number flows, ranging from oil and water to molten polymers (Charles & Redberger 1962; Everage 1973; Hasson & Nir 1970; Minagawa & White 1975; Southern & Ballman 1973; White & Lee 1975; Williams 1975; Yu & Sparrow 1969).

It is necessary to reconcile the existence of a continuum of solutions with the experimentally observed unique configuration. Up to now, explanations have been based on the 'viscous-dissipation principle', which says that the flow chooses an interface which in some sense minimizes viscous dissipation for a given flow rate, or, equivalently, maximizes the volume flux for a given pressure gradient (Everage 1973; Southern & Ballman 1973; Williams 1975; MacLean 1973).

This requires the minimization of

$$\int_{\Omega_1} \{\tfrac{1}{2}\mu_1(\nabla u)^2 - Gu\} + \int_{\Omega_2} \{\tfrac{1}{2}\mu_2(\nabla u)^2 - Gu\},$$

where $G$ is the given pressure gradient, $\mu_i$ and $\Omega_i$ are the viscosity and region respectively of fluid $i$ ($i = 1, 2$), and $u$ is the axial velocity. The union of $\Omega_1$ and $\Omega_2$ is the cross-section $\Omega$ of the pipe. The areas of $\Omega_1$ and $\Omega_2$ are prescribed. The integral represents minus half of the dissipation. The axial velocity $u$ must vanish on the boundary of the pipe and be continuous across the interface. The continuity of shear stress is satisfied automatically as a natural boundary condition. We note that the integral must be minimized not only with respect to $u$, but also with respect to the possible choices of $\Omega_1$ and $\Omega_2$.

For a pipe with general cross-section, the viscous dissipation principle does not have a solution with a smooth interface (see Lurie, Cherkaev & Fedorov 1982 and references therein). Rather, minimizing sequences lead to patterns involving layered structures with thinner and thinner alternating layers of the two fluids. In the limit, this leads to a region that is not filled by either fluid, but by an anisotropic mixture. It can be shown (Tartar 1975; Raitum 1978, 1979) that a modified formulation of the problem allowing such anisotropic mixtures does lead to the existence of minimizers. In reality, surface tension would not permit the formation of layered composites, and emulsions might form instead (if the viscous-dissipation principle is correct).

For a circular pipe, a classical solution does exist. Let $\Omega$ be a circular disk. Following an idea of Everage (1973), let $u_0$ be the solution of $\Delta u_0 = -G$, such that it vanishes on the boundary of $\Omega$. We put $u = u_0/\mu + \bar{u}$. Then the integral above is equal to

$$\int_{\Omega} \nabla u_0 \cdot \nabla u - Gu + \int_{\Omega_2} \tfrac{1}{2}\mu_2(\nabla \bar{u})^2 - \frac{1}{2\mu_2}(\nabla u_0)^2 + \int_{\Omega_1} \tfrac{1}{2}\mu_1(\nabla \bar{u})^2 - \frac{1}{2\mu_1}(\nabla u_0)^2,$$
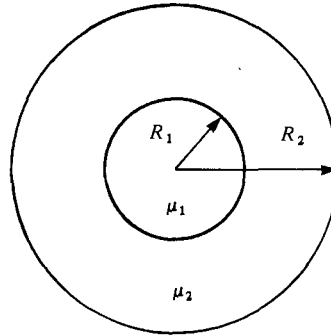
FIGURE 2

where $\Omega_2$, say, is the region occupied by the more viscous fluid. The first of the expressions is zero. The term

$$\int_{\Omega_2} \frac{1}{2\mu_2}(\nabla u_0)^2 + \int_{\Omega_1} \frac{1}{2\mu_1}(\nabla u_0)^2$$

is maximal if and only if $(\nabla u_0)^2$ takes its smallest values in $\Omega_2$; i.e. if $\Omega_2$ is a disk in the centre of the pipe. In this case each boundary of $\Omega_1$ is a line on which $u_0$ is constant. If we then choose $\bar{u} = 0$ in the outer region, and $\bar{u} = $ constant in the inner region such that $u$ is continuous, then the continuity of velocity and shear stress across the interface are satisfied. The expression

$$\int_{\Omega_2} \tfrac{1}{2}\mu_2(\nabla \bar{u})^2 + \int_{\Omega_1} \tfrac{1}{2}\mu_1(\nabla \bar{u})^2$$

becomes zero, which is its minimal value. Hence the minimizer of viscous dissipation is the concentric configuration with the more viscous fluid at the core.

This result appears to agree with experiments. Our question is: how valid is the viscous-dissipation principle? One way to find out is to do a stability analysis for the circular pipe to see if the configuration preferred by the viscous-dissipation principle turns out to be stable.

## 2. Numerical calculations

Following Hickox (1971), we consider a linear stability analysis for the circular pipe where the basic flow is the Poiseuille flow with a concentric interface (figure 2). Fluid 1 is at the core, fluid 2 encapsulates fluid 1. We superimpose an infinitesimal disturbance $(u, v, w, p) \exp \mathrm{i}(-\alpha c t + \alpha z + n\theta)$. We use a Chebyschev-polynomial expansion in the radial direction (Orszag & Kells 1980). The problem is then an eigenvalue problem for $c$, given all the other parameters. If the sign of the imaginary part of $c$ is positive, then the flow is unstable to small disturbances.

The particular case of the long-wave limit, $\alpha \times$ Reynolds number $\to 0$, and the thinner fluid at the core, was studied by Hickox (1971) and was shown to be unstable. This supports the viscous-dissipation principle, but Hickox did not look at the case where the thicker fluid is at the core to see if that would be stable.
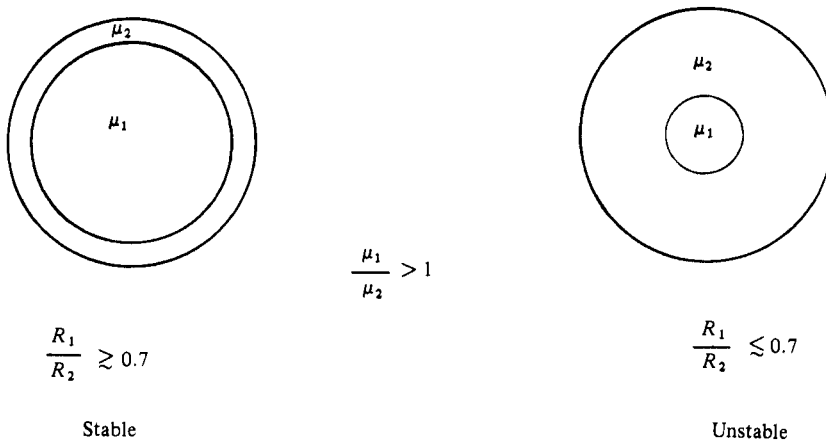
$$\frac{\mu_1}{\mu_2} > 1$$

$$\frac{R_1}{R_2} \gtrsim 0.7 \qquad\qquad\qquad\qquad \frac{R_1}{R_2} \lesssim 0.7$$

Stable                                                                Unstable

FIGURE 3

## 3. Results

The eigenvalue that determines instability is an interfacial one in the sense that it is neutrally stable when the two viscosities are equal. This situation is not identical with the one-fluid flow because of the extra conditions at the interface. Yih (1967) found similar results when he looked at plane Couette flow with a flat interface with the long-wave approximation.

Our range of parameters is the following: viscosity ratio $\mu_1/\mu_2$ from 0.2 to 8, dimensionless wavelength of axial disturbance $\alpha R_2$ from 0.1 to 10, reference Reynolds number $R_2 W/\nu_2$ from 0 to 1000, where $W$ is the 'velocity scale' defined by $GR_2^2/4\,\mu_2$, normalized to 1 in the following, and $\nu_2$ is the kinematic viscosity of the outer fluid. The density is taken to be 1.

First, we found that the configuration with the thin fluid at the core is unstable. This extends Hickox's long-wave results and agrees with the viscous-dissipation principle. Secondly, when the thick fluid is at the core, stability depends on the radius ratio $R_1/R_2$. This shows that the viscous-dissipation principle is not always true. The dependence of the stability on the radius ratio is qualitatively similar to Yih's results, where stability depends on the depth ratio of the two fluids. Figure 3 shows an example of what we found at Reynolds number 100, $\alpha R_2 = 1$.

Figure 4 is a graph of the imaginary part of $c$ versus viscosity ratio for $Re = 100$, $\alpha R_2 = 1$, $R_1/R_2 = 0.7$. Numbers next to the curves denote azimuthal mode numbers. The dark points on the curves show our computed values and the dashed lines are interpolants. At any radius ratio, high azimuthal modes are unstable, but the magnitude of Im $(c)$ decreases asymptotically with the mode number. Here mode 5 becomes positive in the inset. For $R_1/R_2 \gtrsim 0.7$, the curves sink below the Im $(c) = 0$ axis for $\mu_1/\mu_2 > 1$, while higher modes are weakly unstable. For $R_1/R_2 \lesssim 0.7$, all the modes are unstable, yielding the results in figure 3. Figure 5 is a graph of Im $(c)$ versus viscosity ratio at $Re = 100$, $R_1/R_2 = 0.8$, $\alpha R_2 = 10$. When $\alpha R_2 = 1$ the region of stability is $\mu_1/\mu_2 > 1$, but for $\alpha R_2 = 10$ the modes are mostly unstable. The flow is unstable to short waves in $z$ and $\theta$ when surface tension is 0. This agrees with the analysis of Hooper & Boyd (1983), who consider the linear stability of an unbounded Couette flow. The two fluids occupy each half-plane. Their analysis is relevant locally at any interface with a viscosity jump and predicts instability for short-wave disturbances. This is in contrast with one-fluid flows, where viscosity acts to dampen
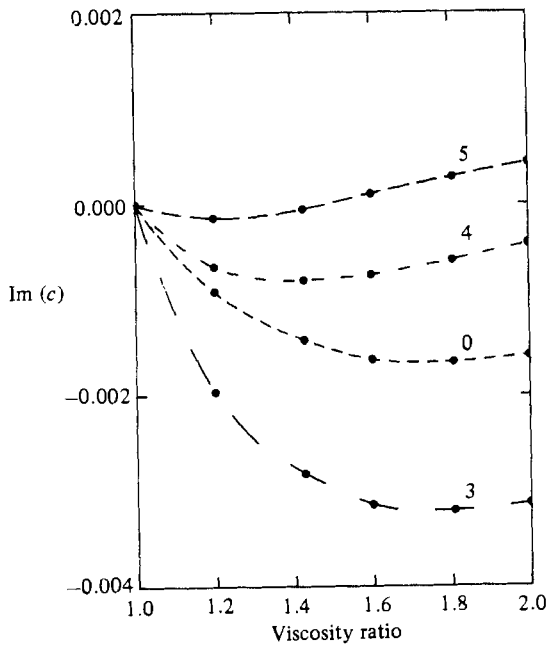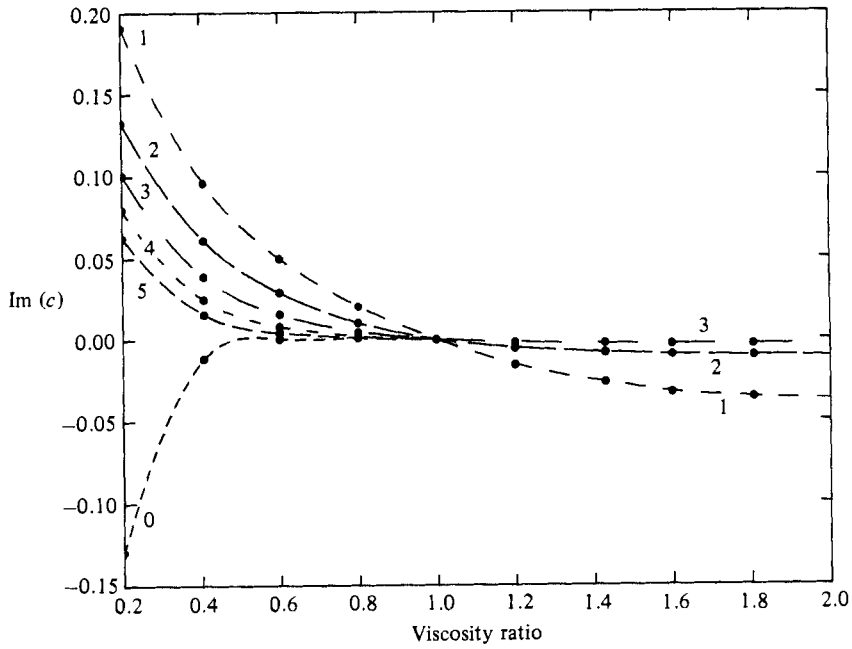
FIGURE 4. $Re = 100$, $\alpha R_2 = 1$, $R_1/R_2 = 0.7$. The magnification on the bottom displays behaviour for $1 \leqslant \mu_1/\mu_2 \leqslant 2$.
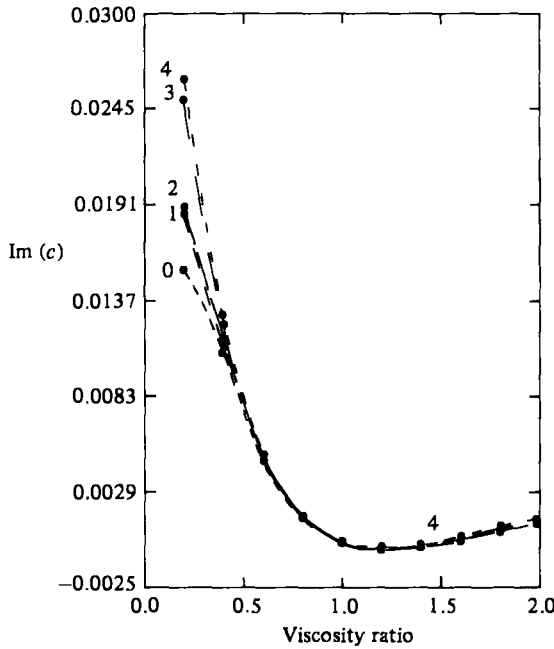
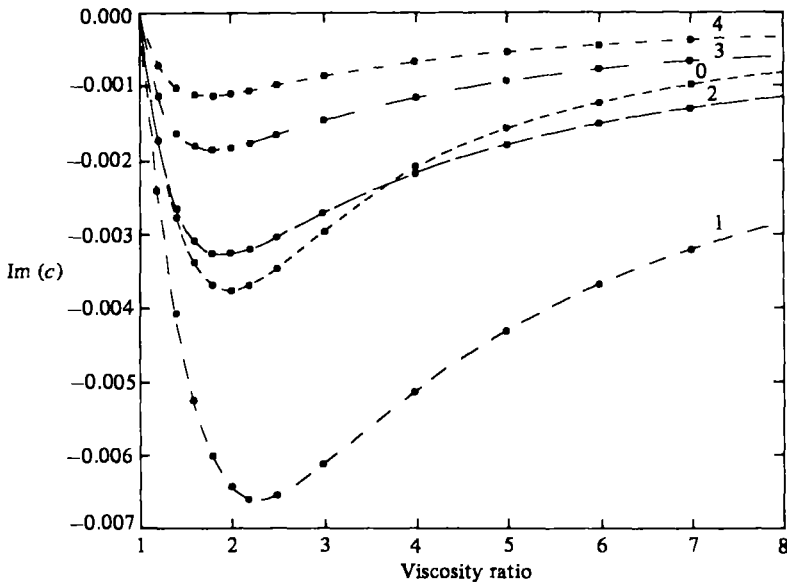FIGURE 5. $Re = 100$, $\alpha R_2 = 10$, $R_1/R_2 = 0.8$.



FIGURE 6. $Re = 100$, $\alpha R_2 = 1$, $R_1/R_2 = 0.9$.

short waves. Hooper & Boyd show that surface tension is effective in dampening short waves.

Thirdly, when the viscosity ratio is large, the response changes only gradually. Figure 6 is a graph at $R_1/R_2 = 0.9$, $Re = 100$, $\alpha R_2 = 1$. The imaginary part of $c$ is not sensitive to changes of the ratio $\mu_1/\mu_2$; for example from $\mu_1/\mu_2 = 6$–7. This behaviour has been mentioned in some experiments (Everage 1973).
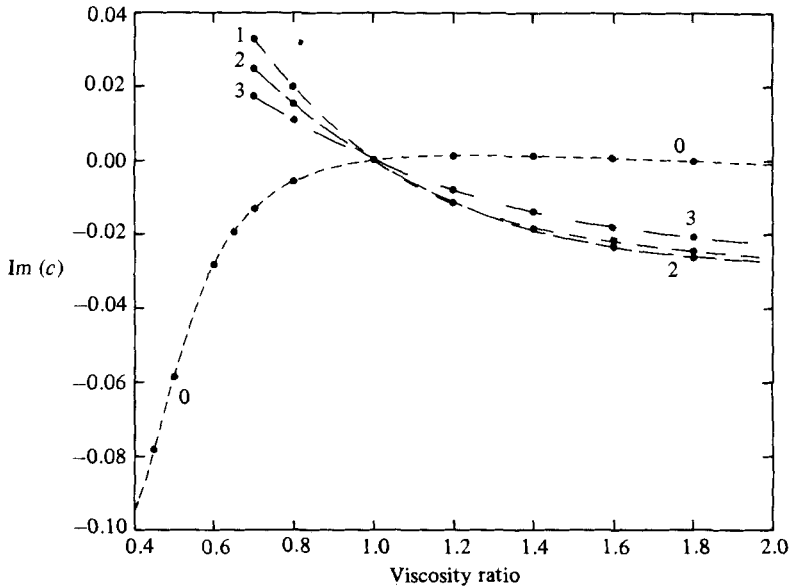
FIGURE 7. Imaginary part of $c$ versus viscosity ratio $Re = 1000$, $\alpha R_2 = 1$, $R_1/R_2 = 0.8$.

Fourthly, as the Reynolds number increases, stability is lost. Figure 7 is a graph of $Re = 1000$, $R_1/R_2 = 0.8$, $\alpha R_2 = 1$. When $Re = 100$, the region of stability is $\mu_1/\mu_2 > 1$, but when $Re = 1000$ the region of stability is reduced to $\mu_1/\mu_2 \gtrsim 1.8$. Detailed numerical results and calculations are described in Joseph, Renardy & Renardy (1983).

## 4. Conclusion

Our results show that the restricted version of the viscous-dissipation principle described in this paper is not always true. However, figure 3 indicates that there is some truth to the idea that the thin fluid tends to lubricate the wall. The flow with the thin fluid at the centre is very unstable to long waves. The flow with small amounts of thin fluid outside is stable to long waves. The flow with large amounts of thin fluid outside is weakly unstable to long waves. Flows with thin fluid outside are weakly unstable (growth rates tend to zero) to short waves and are probably stabilized by surface tension.

The basic feature that the less-viscous fluid tends to lubricate the wall is also found in plane Couette and Poiseuille flow (Yih 1967) and in the flow between rotating cylinders (Renardy, Y. & Joseph 1983). In the latter case, a lubrication layer on either cylinder turns out to be stable, and the stabilizing effect of viscosity stratification can even overcome a destabilizing density difference.

A natural question arises as to which flows would replace the concentric flow when it becomes unstable. If surface tension is important, then short waves are stabilized, and the dynamics of the problem should be governed by long and order-1 waves. If periodic boundary conditions are imposed in the streamwise direction, then there are a finite number of such modes, and techniques of bifurcation theory are applicable. The simplest patterns that can arise are travelling interfacial waves arising from a Hopf bifurcation. This has been conjectured by Yih (1967), and a rigorous proof is

given in a forthcoming paper (Renardy, M. & Joseph 1984). If surface tension is absent, then there is an infinite number of unstable modes with arbitrarily short wavelengths. The usual techniques of bifurcation theory do not apply to this type of situation. We believe that this situation is a possible mechanism for the formation of emulsions.

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